Uncertainty theory and complex systems scenarios analysis

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1. Introduction

Coupled socio-economic-physical systems are the most complex of all. They involve vastly different types of uncertainties, including those related to human behavior and those of a scientific nature, such as Earth system feedbacks. These uncertainties cannot be dealt with in a uniform manner. The integrated assessment community, as an evolution of the system analysis community, has been looking for decades to go beyond subjective probabilities to model uncertainty.

The recent history of climate change prospective illustrates the point. A critical limitation of the Intergovernmental Panel on Climate Change (IPCC) special report on emissions scenarios (SRES, 2000) is that the scenarios are symmetrically presented, and no judgment is offered as to the preference for any of them. Admittedly, the political sensitivity of alternative plausible futures played a significant role in this reluctance to quantify. Yet this lack of specificity was explained by a deeper theoretical issue about uncertainty and probabilities. We need first to remember the difference between two kinds of situations, hereafter named Risk and Uncertainty.

In this text, Risk describes a situation in which all the possible outcomes are known in advance, and each has a probability weight. Uncertainty describes a more general class of situation in which, while all possible outcomes are known, probabilities may or may not available. Higher order of ignorance such as incompleteness and surprises and others discussed in Smithson (1988) will not be considered here.

This fundamental difference between risk (when probabilities are available) and uncertainty (when they may not be) is crucial to understand how, if not why, the IPCC declined to put probability weights on scenarios. This difference has been stressed by social scientists at least since Knight (1921) and Keynes (1921). The dominant paradigm in the second half of the last century to model uncertainty was to use subjective probabilities, yet the IPCC position is only the last of a long history of challenges to this dominant paradigm, exemplified by Allais in 1953.

This conceptual distinction has been common in decision analysis for a long time. But until recently the distinction did not matter in practice for technical reasons: IPCC wrote that there was no widely accepted mathematical methods to deal with Uncertainty that is not Risk in integrated assessment. Realizing at the same time that future emissions are definitely an uncertainty situation, it was a legitimate way out of a political minefield when Grübler and Nakicenovic (2001) stated "the concept of probabilities as used in the natural sciences should not be imposed on the social sciences. Scenarios [...] are radically different." These leading IPCC authors' point of view was important in the refusal to assign probabilities to the new scenarios.

A few climate change integrated assessment using alternative uncertainty formalisms appeared in the late nineties. Leimbach (1996, 1998) worked on fuzzy optimization for climate strategies. Welsch (1995) examined greenhouse gas abatement under ambiguity (Ambiguity is used by economists to name uncertainty that is not risk). While fuzzy techniques appear to be used in models of impact and land-use changes, so far they remain absent from energy policy decision analysis. There is no standard alternative to subjective probabilities in integrated assessment.

Uncertainty theories have deep historical roots, but it is only recently that they have demonstrated a practical engineering value. The theory of evidence using belief functions was published by Shafer in 1976, and possibility theory (based on fuzzy sets) was proposed as a complete alternative to probability theory by Zadeh in 1978. The reason why these theories emerged only in the computer age is probably because they are better suited to numerical computations than to algebraic analysis.

Another problem is that there may be too many mathematical methods to deal with non-probabilistic uncertainty. Reactions to this last issue may be dogmatic (only one theory should be used), eclectic (each theory is good in its domain of validity) or unifying (each theory correspond to a projection of a more general notional theory). This paper defends the unifying conception.

2 The simplest theory of uncertainty theory: Belief functions

This section introduces a very simple theory of uncertainty, known as the Dempster-Shafer theory of evidence. Because we suggest to use this theory to analyze uncertainty in complex systems scenarios, we will consider the question of belief about which one of the forty SRES scenarios describes the future. So $\Omega = S40$, and similarly the IPCC sets of six and four scenarios will be denoted *S6* and *S4*.

The theory represents beliefs as a [0, 1]-valued function on the power set of possible futures $\wp(\Omega)$. One way to do so is to consider a probability distribution *m* (called a basic belief assignment) defined on all subsets of *S40*. For any subset of *S40*, for example *S6*, the strength of the belief that the future world is in *S6* is given by the formula $\text{bel}(S6) = \Sigma\, m(A)$, for all $A \subseteq S6$.

The crucial point of all theories of uncertainty is that support for *S6* is not equal to the absence of evidence against *S6*, so that bel(S6) is less than $1 - \text{bel}(S40 - S6)$. The latter is called the plausibility of *S6* , it can also be defined as $\text{pl}(S6) = \Sigma\, \text{bel}(A)$, for all A such that $A \cap S6 \neq \emptyset$. Beliefs may be low, but plausibility high: this is the key to quantifying uncertainty in scenarios.

An extreme case of this is that belief functions allow for proper vacuous priors: instead of the uninformative interpretation of equi-probability over *S40* , the belief is defined by $m(S40)=1$, $m(E)=0$ otherwise for any $E \not\subset S40$. In this situation, the plausibility of any subset is 1, but the belief of any proper subset is zero.

For any subset *E*, the basic belief *m*(*E*) represents the amount of non-specific weight of evidence given to *E*. Non-specificity means that the information can not be pointed further to any element in *E*. This allows to make judgments about classes of scenario, such as "high and rich population scenarios are not likely".

bel(*E*) represents the strength of the belief that the real state of the world is described by a scenario in *E* . Beliefs are super-additive: $\text{bel}(A \cup B) \geq \text{bel}(A) + \text{bel}(B)$ even when $A \cap B = \emptyset$. Note that bel(∅) represents the belief that the real state of the world is not in *S40*. It is normalized to zero if one assumes from the start that *S40* represents the complete list of possible futures.

In sum, any of three following elements define beliefs:

A basic belief assignment, that is a probability distribution *m* defined on all subsets of *S40*.

The belief function, that is $\text{bel}(E) = \Sigma\, m(A)$, for all $A \subseteq E$

The plausibility function; that is $\text{pl}(S6) = \Sigma\, \text{m}(A)$, for all A such that $A \cap S6 \neq \emptyset$

3 The unity of uncertainty theories

Evidence theory (beliefs functions) is only one of a large number of superficially different approaches to uncertainty. This section outlines the main links of an unified view of uncertainty theories, referred to as Imprecise Probabilities theory by Walley (1991).

The first link between all uncertainty theories is the difference between "support for an idea *E*" and "absence of evidence against *E*". Support for *E* , denoted $\underline{P}(E)$, relates to the notion of belief, necessity or lower probability. The absence of evidence against *E*, that is $\overline{P}(E) = 1 - \underline{P}(\Omega - E)$, corresponds to the concepts of plausibility, possibility or upper probability. Situations of risk, and only those, are characterized by the identity of the two notions: $P(E) = 1 - P(\Omega - E)$. In all other uncertainty situations, the two numbers do not add up to unity.

The second link is the correspondence between qualitative (*A* is more believed than *B*) and quantitative (P(A) = 0.2 is greater than P(B) =0.1) approaches. Trivially each quantitative approach represents a comparative ordering of beliefs. The converse is true in many interesting cases: given reasonable axioms on a partial order relation defined on subsets, it is often possible to find a real function representing the relation. This correspondence is useful because elicitation of beliefs is easier using the qualitative approach.

The third link is the canonical Boolean algebra homomorphism between set theory and propositional calculus. The union of two subsets corresponds to the disjunctive operator OR, the intersection corresponds to the conjunction AND, and inclusion corresponds to implication. This correspondence is useful because computer implementations are often easier using formal logic. It uses a small finite set of symbols to build an infinite number of propositions, instead of using a very large finite set of states of the world. In this way, propositional calculus allows to formalize, beyond situations of uncertainty, situations incomplete information.

This correspondence also leads to two other important remarks. First, there are many different operators to connect propositions. There are as many different of ways to combine beliefs. Second, the notion of granularity refers to how the [0, 1] interval is divided to grade beliefs. Logic divides uses just two points, while most other theories use the whole interval. It has been suggested that in practice, dividing the interval into 5 to 7 levels of confidence is enough.

The fourth link is the correspondence between finite, countable and continuous versions of the theories. With respect to these differences, probability was been unified by measure theory a long time ago. Similarly, belief functions defined in the previous section can be regarded as Choquet capacities of order infinity defined for a finite Ω.

Fifth, there are several equivalent approaches to belief functions: we have already seen basic belief assignment, belief function, plausibility functions and Choquet capacities (totally monotone set functions). There are many more. In particular, Dempster introduced the theory using many-valued mappings, while Shafer focused on representing beliefs as the combination of elementary pieces of unspecific information. These latter approaches are important, since working directly with the power set of the states of the world is not computationally tractable.

Six, there are inclusion links between uncertainty formalisms.

i) A probability distribution is a belief function where the basic belief assignment m is non-zero only on singletons. A possibility distribution is a belief function where m is non-zero only on a nested family of subset.

ii) Uncertainty is often represented using a convex family of probability distributions called a credal set. Any belief function canonically defines a core family of probability distributions F by: $p \in F$ if and only if for any event E, $\text{bel}(E) \leq p(E) \leq \text{pl}(E)$. Any probability in the core is is some sense compatible with beliefs, but it does not represent them completely. The core is never empty, since it contains the probability defined by $p(\omega) = \Sigma\, m(A) / |A|$, for all $A \subseteq \Omega$ containing ω. This distribution (called the pignistic probability) spreads equally unspecific information across all possible states, it represents one way to derive rational betting rates under uncertainty.

The belief function can be recovered as the lower envelope of its core, but the lower envelope of a credal set is not necessarily a belief function.

iii) The important hindsight is that this family of probability distribution needs not to be probabilized, so second-order probabilities are not needed, only the extremal points matter. Given a credal set F , for any loss function l: $\Omega \rightarrow \Re$, one can define its lower expectation $\underline{E}(l) = \inf E_p(l)$, for all p in F. Reciprocally, given an arbitrary functional $\underline{E}(l)$, it is possible to define a family F of probability distributions p that verify $\underline{E}(l) \leq E_p(l)$, for all l . This establishes a one-to-one correspondence between convex sets of probability distributions and affinely superadditive lower expectations (Cozman, 1999). This is the correspondence between Walley's (1991) theory of coherent lower previsions (i.e. expectations) and the theory of credal sets.

To summarize this hierarchy, coherent lower previsions are the same mathematical beast as credal sets. They generalize belief functions. Belief functions unify probabilities and possibilities. All these correspondences explain at the same time a superficial diversity of uncertainty theories and a deeper unity. Still there remains some residual conflicts between theories which needs to be addressed.

First, there are two different paradigms on the question of decision making under uncertainty. One assumes that preferences define a total order, so there is always one best choice, maybe given by the precautionary principle or by maximizing Choquet expected utility. The other paradigm assumes that under uncertainty there is only a partial order, then there is only a set of maximal choices. The unifying answer to this may be to extend the notion of rationality. It is already accepted that different individuals have different preferences with respect to risk and ambiguity, so we don't see why everybody should follow the same decision model under uncertainty.

Second, there are significant differences in the nature of the uncertainty being represented. For example an uncertainty theory dealing with statistical evidence would have different semantics than a theory dealing with opinions. In particular, Dempster-Shafer rejects the interpretation that there is one unknown probability distribution. The unifying point of view is that these differences should lead to different operators for combining the mathematical objects representing uncertainty.

4. Analysing uncertainty in complex systems: Quantifying prospective scenario sets

After having defended the unifying point of view at the foundations level, we have to fall back to the eclectic point of view when it comes to applications. Probabilities work very well for most of engineering issues today, and possibilities also have their applications. To which kind of uncertainty analysis problems would Dempster-Shafer theory be useful? I propose to use belief functions to quantify uncertainty for long-term prospective scenarios, along the following lines:

Quantifying uncertainty: Uncertainty on *S40* can be quantified by considering evidence from IPCC reports and from the larger database of all scenarios submitted during the SRES open process. Obviously, the belief function on *S40* will not be represented by tabulating $m(A)$ for all $A \subseteq S40$. Yet beliefs can be transferred from *S40* onto the smaller *S4* and *S6* scenario sets and given explicitly for *S4*. Following standard practice, beliefs will be defined by and represented as the combination of elementary pieces of unspecific information about scenarios, such as the assumption that high per capita income and high population growth are not likely.

Probabilizing the scenario sets: Beliefs allow to explain the notion of a core of probability distributions. Picking up distributions from the core is illustrative. Different ways to parameterize value judgments by norms, distances and other weight factors imply different rational probability distributions for a given belief function. Upper and lower probabilities can be found for *S4* and *S6*, as well as the pignistic probability distribution on each of these scenario sets.

Reducing the number of scenarios: For each of the 64 scenario sets obtained by deleting various scenarios from S6, the loss of information, the loss of plausibility and of credibility compared to S6 and S40 can be explored. This will make explicit how well S4 and S6 represent the uncertainty on S40. One can quantify (in bits) the information gap between S4 and S6. This task will identify which scenarios to remove from S6 and S4 in a way that minimize the information loss. It defines scenario sets S6-3, S6-1 and S4-1, the first removing 3 scenarios from S6, the second removing 1 scenario from S6, and the last removing one scenario from S4.

Extending the scenario sets: One can quantify which scenario from S40 would be the most interesting to add to S6 or to S4. Interest will be defined as the overall plausibility of the scenario set, the credibility of the range it covers and the information content. The same questions as in the previous tasks will be answered: What are the probabilities of the augmented scenario sets? What is the gain in information, plausibility and credibility? This will lead to augmented scenario sets S6+1 and S4+1.

Comparing SRES with optimal the scenario subsets: There are 91390 ways to pick up four scenarios from S40. Only a few are really interesting: those maximizing plausibility, credibility and information content. The constructive IPCC method to define S4 and S6 in no way guarantees optimal results. I will study the role of rationality in the open process, and then assess its efficiency by comparing S4 with optimal quadruples of scenarios from S40, and also comparing S6 with an augmented S4+2 obtained by optimally adding two scenarios to S4.

To summarize, the principles of maximizing entropy, plausibility and credibility allow to define what is a interesting scenario set, which fairly represent the diversity of results in a maximally informative way.

5. Conclusion

This conclusion outlines five advanced questions relevant for future research on uncertainty.

The dynamics of uncertainty. The counterpart of stochastic dynamic programming remains to be explored in Uncertainty situations. A central practical question is how to reduce the curse of dimensionality over time by allowing time to flow on a graph (a network) instead of a tree. The fuzzy horizon effect is commonly used in rendering landscape on a computer. An information kinetic counterpart of this could be that probabilistic uncertainty decays into unspecific uncertainty over time. This allows decreasing conflict and effectively reuniting separated threads of time. For example, it may be initially known that a variable can be A with probability p, and B with probability 1-p ; but in the long run it is only known that the variable is either A or B. This replaces a two-cases situation by a single case.

Knowledge-based integrated assessment modeling. This research also paves the way for bridging traditional system analysis with a radically new approach to integrated assessment: defining the model purely as structure on the belief function. Theoretically, this approach encompasses traditional modeling analysis: believe only in the couples (parameters, results) that are given by the model. But this is more general, it allows for uncertain relationships. Early applications of this new approach can be found in robust modeling, fuzzy systems or constraint-based programming. One example of this approach is the NetWeaver method by Parker (2001).

Experimental design. Sampling is important for integrated assessment, as some coupled models of earth dynamics take months for a single run. Uncertainty theories are promising a fresh approach of these issues addressed at length in the Bayesian framework. Given a model, a scenario set should explore all qualitative regions of the outcome space, but there are two problems with this. First, the number of attractors for the dynamic system may be larger than the number of scenarios. Second, the outcome space is computed by the model, so even assuming a small number of well-defined qualitative results, it is not possible to know how many beforehand. I conjecture that using the model to transfer uncertainty from parameter space to the outcome space can lead to robust scenario sets.

Decision-making: Because they represent socio-economic systems, with thinking agents, scenarios should integrate some sense of rationality with respect to the decision making problem under uncertainty. The problem is that non-specificity does not necessary allow to single out just one optimal course of action, as we have seen above the question of rationality, coherence and precaution is renewed in the non-bayesian framework. Our point of view is that scenario-based analysis can take advantage of having only a partial ordering of actions. Because scenario analysis is not prescriptive, it is enough to reject dominated actions, and to describe a maximally preferred course of action avoiding sure loss.

Applications: The biggest challenge of all is to provide software to help people to take better decisions. It is necessary to provide tools for constructing belief functions by fusion of a body of elementary opinions (such as possibility statement or comparative probability statements). This must be completed with means to analyze uncertainty and communicate it into simplified but adequate terms. Finally, there is a need to optimize choices or at least find maximally preferable actions.

Overall, recent mathematical developments on uncertainty analysis translate into a general quantitative method for building more plausible and credible scenarios sets. Models are often used in politically charged situation, with the goal of providing a cold analysis. In a scenario building methodology, the informal steps represent a major point where non-scientific, vested interest parties can capture the process. Beliefs functions provide a wider mathematical framework for experts to formalize their knowledge and reach consensus.

6. Bibliography

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