The real option with absorbing barrier*

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Abstract

This paper analyzes the theoretical problem of the real option with barrier. It models an investment decision with a double irreversibility concern: investing is irreversible, but waiting runs the risk of loosing the opportunity to invest. The optimal strategy leads to earlier investment when the barrier increases, or when uncertainty decreases. Uncertainty has ambiguous effects on the expected decision time and on the investment probability after N years. Analytical and numerical results also apply to the perpetual American call with a down-and-out barrier on a dividend paying asset.

Keywords: Real option, barrier, irreversibility
JEL: G13, D92, D81

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1 Introduction

Time, risk and irreversibilities are at the core of the real option theory of investment. This theory studies a simple decision problem: when to realize an irreversible investment, given that its cost I is known and constant, whereas its return follows a stochastic process $V(t)$? The central result is that acting as soon as $V(t) \geq I$ does not maximizes the expected discounted benefit. Instead, it is optimal to invest only when $V(t)$ reaches a critical level V^* greater than I.

This paper analyzes the theoretical problem of the real option with barrier. It models an investment decision with a double irreversibility concern: investing is irreversible, but not investing runs the risk to loose everything if one waits too long. More precisely, the investment opportunity is assumed gone forever if $V(t)$ ever reaches some specific barrier level V_m.

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Given that "One bird in the hand is worth two in the bush", the intuitive effect of the barrier is straightforward: it suggests to invest earlier. The substance of this paper is in the mathematical proofs formalizing this idea.

Barrier options have mostly been analyzed in financial pricing models. The problem studied in this paper is formally equivalent to pricing the perpetual American option on a dividend-paying asset with a down-and-out barrier. We did not found this specific kind of option studied in previous literature such as Rich [1994], Gao et al. [2000], Haug [2000], Karatzas and Wang [2000], Zvan et al. [2000].

This is hardly surprising considering that commodity barrier options trading is less than a decade old, and that there are fundamental differences between financial and real options. Financial options are priced using a time-dependent equilibrium equation. Real option theory uses an optimality Bellman equation, stationary because no deadline to invest is posited in the basic model. This stationarity allows to push the analysis further before resorting to numerical computations.

In their seminal real option work, Brock and Stiglitz [1989] did note the possibility to extend the model with a barrier, but did not do it. To the best of our knowledge, this point has not been raised again the subsequent literature, other than to note its mathematical technicality. For example, Dixit and Pindyck [1994] focus on the $V_m = 0$ case all through their book.

This re-visit of real option theory is organized in four parts. Next section defines V^*, the optimal investment threshold and examines its variations with respect to the barrier level V_m. It is shown that V^* decreases smoothly as the barrier increases in $[0,\, I[$, with a vanishing derivative to the left and a discontinuity to the right.

The second section analyzes the influence of uncertainty. It shows that generally V^* increases (and the probability of investing decreases) when volatility increases. This confirms the intuitive result that uncertainty depresses investment at the aggregate economic level. This section also analyzes how the expected time to decision and the investment probability depend upon uncertainty

The third section completes the picture by giving the initial probability densities. Since these involve special functions, controlled numerical approximations are given. They are used to illustrate Sarkar [2000] non-monotonicity result on the probability of having invested after a given duration

The last section discusses examples and critical economic aspects of the model.

2 The critical value for investment with barrier

Assume that V follows a geometric Brownian motion with drift, that can be denoted by $dV = (\rho - \delta)V dt + \sigma V dz$. The process is killed at the lower barrier V_m and at the critical level V^* where investment occurs. The barrier is given assuming that $0 \leq V_m < I$. Since everything is homogeneous in $(V,\, I)$ let us denote $x = V/I$, and $m = V_m/I$. In these normalized variables, the relative investment value x follows a geometric Brownian motion with the same parameters as V, killed at first exit from the interval $[m,\, x^*]$. By hypothesis, both m and x^* are killing barriers, but their payoff differ.

Let $F(V)$ be the value of the option to invest, and $f(x) = F(V)/I$ its counterpart in relative terms. If the upper bound x^* is reached first, then investment occurs. Consequently, at this point the option value is the investment's net benefit, that is $f(x^*) = x^* - 1$. But if the lower bound m is reached first, then the option to invest is irreversibly lost. Consequently, the option value is zero when x hits the barrier m, that is $f(m) = 0$.

This paper assumes that the investment strategy is to keep the option alive as long as x is inside the interval $[m, x^*]$, and to invest as soon as the upper bound is reached. It is economically interesting to ask if that such a strategy allows to maximize the expected discounted gain over all conceivable investment strategies? In the real option literature, that property of global optimality is a well-known result see [Dixit and Pindyck, 1994, ch. 4, appendix C, pp. 128-131] for example.

The decision maker determines the x^* maximizing the expected discounted gain. The lower barrier m being given, the upper bound x^* is determined by solving the Bellman equation for the problem, which characterizes the optimality condition for the investment:

$$\frac{\sigma^2}{2} x^2 f''(x) + (\rho - \delta) x f'(x) - \rho f(x) = 0 \tag{1}$$

The general solution of this equation is $f(x) = C_1 x^{\beta_1} + C_2 x^{\beta_2}$, where C_1 and C_2 are two constants factors, and the β_i are the roots of the fundamental quadratic $(\beta - 1)\beta\sigma^2/2 + \beta(\rho - \delta) - \rho$. Let us note $\beta_1 = \lambda + \Delta$ the root greater than 1, and $\beta_2 = \lambda - \Delta$ the negative root, with:

$$\lambda = 1 + \frac{2(\delta - \rho)}{\sigma^2} \tag{2}$$

$$\Delta = \frac{\sqrt{8\rho\sigma^2 + (2\delta - 2\rho + \sigma^2)^2}}{\sigma^2} \tag{3}$$

When $m = 0$, as in the classical real option model, the lower boundary condition is $f(0) = 0$ so that $C_2 = 0$. Crucially in this paper, the lower boundary condition is $f(m) = 0$. The general solution satisfying it can be written as:

$$f(x) = C_1(x^{\beta_1} - m^{\beta_1 - \beta_2} x^{\beta_2}) \tag{4}$$

The smooth pasting conditions, where x^* is the exercise value, are $f(x^*) = x^* - 1$ and $f'(x^*) = 1$. Together with equation (4), they determine C_1 and x^* that maximize the expected present value of the investment opportunity. Therefore the exercise price x^* is the solution of the implicit equation $\Psi_{\beta_1,\beta_2,m}(x) = 0$, where:

$$\Psi_{\beta_1,\beta_2,m}(x) = x^{\beta_1 - \beta_2}(x(\beta_1 - 1) - \beta_1) - m^{\beta_1 - \beta_2}(x(\beta_2 - 1) - \beta_2) \tag{5}$$

When $m = 0$ case, expression (5) implies that the critical value is $x^* = \beta_1/(\beta_1 - 1)$, as in Dixit and Pindyck [1994].

The implicit equation can also be presented as the equality between a power function and an homographic function, that is $g_{\beta_1,\beta_2}(x) = h_{\beta_1,\beta_2,m}(x)$ with:

$$g_{\beta_1,\beta_2}(x) = x^{\beta_1 - \beta_2}$$

	σ =0.2	σ =0.4	σ =0.6
m=0	2.46 (0%)	4.72 (0%)	8.17 (0%)
m=0.2	2.45 (0%)	4.6 (-3%)	7.67 (-6%)
m=0.4	2.42 (-1%)	4.32 (-8%)	6.88 (-16%)
m=0.6	2.34 (-5%)	3.88 (-18%)	5.84 (-29%)
m=0.8	2.16 (-12%)	3.21 (-32%)	4.45 (-46%)
m=1.	1. (-59%)	1. (-79%)	1. (-88%)

Table 1: Sensitivity of the optimal investment criteria x^* to volatility σ and barrier m, with a 1% per year positive trend.

$$h_{\beta_1,\beta_2,m}(x) = m^{\beta_1-\beta_2} \frac{x(1-\beta_2)+\beta_2}{x(1-\beta_1)+\beta_1}$$

The optimal value of x^* cannot be given analytically. Numerical root-finding algorithms work well, since the implicit equation defining x^* is smooth and well behaved, and the root lies in the interval $[1, \beta_1/(\beta_1-1)]$ as we will see now.

Proposition 1 *The optimal x^* is lower with a barrier than without.*

One has to show that x^*, root of Ψ, lies within the interval $[1, \beta_1/(\beta_1-1)]$. Since Ψ is a continuous function of x, it is enough to check that it has opposite sign at the extremities of this interval.

On one hand, $\Psi(1) = m^{\beta_1-\beta_2} - 1$ is negative since $m < 1$.

On the other hand, $\Psi(\beta_1/(\beta_1-1)) = m^{\beta_1-\beta_2}(\beta_1-\beta_2)/(\beta_1-1)$ is positive since $\beta_1 > 1$ and $\beta_2 < 0$.

Therefore, Ψ has a root within the open interval.

Q.E.D.

That property is economically intuitive. It states that when there is a barrier $m > 0$, it is optimal to invest sooner than if there was no risk of loosing the option, i.e. $m = 0$.

For numerical application, [Dixit and Pindyck, 1994, page 153], choose $\rho = 0.04$, $\delta = 0.04$ and $\sigma = 0.2$ at annual rates as sensible order of magnitudes. Table 1 illustrates numerically the effect of the barrier upon the exercise price, considering the case $\rho = 0.04$, $\delta = 0.03$, with σ between 0.2 and 0.6, and m between 0 and 1. With this parametrisation the expected trend $\rho - \delta$ is one percent per year.

Table 1 illustrates results that will be demonstrated formally below. The exercise price x^* is a decreasing function of the barrier m. Comparing the first two rows with any other two consecutive rows illustrates that $\partial_m x^*$ vanishes when $m \to 0$. Examining the last row illustrates that only $x^* = 1$ solves $\Psi_{\beta_1,\beta_2,1}(x) = 0$. There is a possible discontinuity in x^* when $m \to 1^-$, which is also apparent in the table.

Proposition 2 *The exercise value x^* regarded as a function of the barrier m:*
i) $x^*(m)$ is monotonously decreasing with m
ii) $x^*(m)$ is indefinitely smooth on]0,1[and always greater than ρ/δ
iii) $x^*(m)$ is continuous in $m = 0$, with a vanishing derivative

Proof:

i) $x^*(m)$ is decreasing:

Consider a given m and x^* so that $g_{\beta_1,\beta_2}(x^*) = h_{\beta_1,\beta_2,m}(x^*)$. Consider a given $n > m$. To show that the root of $g_{\beta_1,\beta_2}(x) - h_{\beta_1,\beta_2,n}(x) = 0$ is less than x^*, we check that the left hand side of this equation changes sign on $[1, x^*]$.

Since $n/m > 1$, on one hand:

$$g_{\beta_1,\beta_2}(x^*) - h_{\beta_1,\beta_2,n}(x^*) = g_{\beta_1,\beta_2}(x^*) - (n/m)^{\beta_1-\beta_2} h_{\beta_1,\beta_2,m}(x^*) < 0$$

On the other hand,

$$g_{\beta_1,\beta_2}(1) - h_{\beta_1,\beta_2,n}(1) = 1 - n^{\beta_1-\beta_2} > 0$$

Because $g_{\beta_1,\beta_2}(x) - h_{\beta_1,\beta_2,n}(x)$ is continuous, its root lies in the interval $[1, x^*]$.

ii) $x^*(m)$ is indefinitely smooth on]0,1[:

To set up the implicit function theorem with $\Psi_{\beta_1,\beta_2,m}(x) = 0$, consider a couple (x^*, m) in the interior of the set $A = [1, \beta_1/(\beta_1-1)] \times [0, 1]$, such that $\Psi_{\beta_1,\beta_2,m}(x) = 0$. Since Ψ is infinitely smooth, all we have to do is check that $\partial_x \Psi_{\beta_1,\beta_2,m}(x) \neq 0$, to show that locally x^* is a infinitely smooth function of m. When this is the case, the derivative is

$$\partial_m x^* = -\frac{\partial_m \Psi}{\partial_x \Psi} = \frac{x(2x^2\delta + 2\rho - x(2\delta + 2\rho + \sigma^2))}{2m(-1 + x)(x\delta - \rho)}$$

Given that:

$$\partial_x \Psi_{\beta_1,\beta_2,m}(x) = \frac{m^{\beta_1-\beta_2}(x-1)(\beta_1-\beta_2)(x(\beta_1-1)(\beta_2-1) - \beta_1\beta_2)}{x(x(\beta_1-1) - \beta_1)}$$

in the interior of A, the partial derivative can be zero only when $x = x^\dagger$, with $x^\dagger = \beta_1\beta_2/(\beta_1-1)/(\beta_2-1) = \rho/\delta$.

But if there was a value $m^\dagger$ in $]0, 1[$ so that the corresponding $x^*(m^\dagger) = x^\dagger = \rho/\delta$, then since x^* decreases with m, for some $m^\ddagger > m^\dagger$ we would have $x^*(m^\ddagger) < \rho/\delta$. This would therefore ensure the existence of $\partial_m x^*$ around $m^\ddagger$. But then, given its explicit formula above, this derivative would then be positive, which is not possible.

Consequently, for any m, we are sure that $x^* > \rho/\delta$ and the partial derivative is non zero.

iii) $x^*(m)$ is continuous in $m = 0$, with a vanishing derivative:

Since $x^*(m)$ is decreasing, it has a limit when $m \to 0$. Call this limit l. We have to show that $l = \beta_1/(\beta_1-1) = x^*(0)$.

When $m \to 0$, the function $g_{\beta_1,\beta_2}(x^*(m))$ has a limit $l^{\beta_1-\beta_2}$ which is non-zero. Consequently function $h_{\beta_1,\beta_2,m}(x^*(m)) = -g_{\beta_1,\beta_2}(x^*(m))$ has a non-zero limit.

The numerator of h, that is $m^{\beta_1-\beta_2}(\beta_2+(1-\beta_2)x^*)$, has a limit zero when $m \to 0$, since $\beta_1 - \beta_2 > 1$ and x is bounded (proposition 1). Therefore, the denominator of h must also have zero as its limit, so that $l(1-\beta_1) - \beta_1 = 0$.

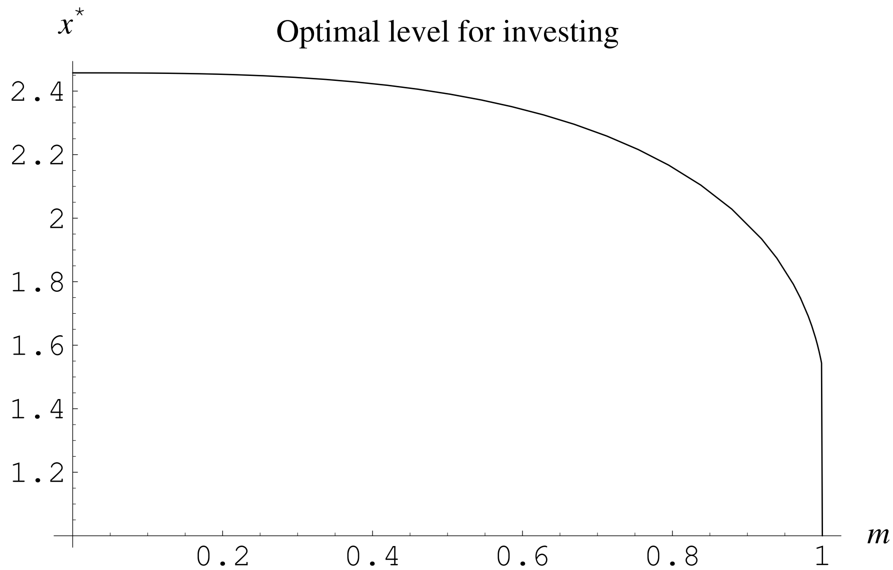

Figure 1: The optimal exercise level x^* as a function of the barrier m.

Having shown that $x^*(m)$ is continuous in $m = 0$, the derivative limit is indeterminate 0/0 when $m \to 0$, but l'Hospital rule shows that $x'(m) \to 0$ indeed.

Q.E.D.

Remark that when $m \to 1$, since $x^*(m)$ is decreasing, it has a limit l' also. Since $x^* > \rho/\delta$, the limit will be greater than or equal to ρ/δ. In the economically interesting situation where the trend is positive, we have $\rho - \delta > 0$, therefore $l' > 1$. This implies a discontinuity since direct resolution shows that $x^*(1) = 1$.

Property *iii)* of proposition 2 can be interpreted as saying that the $m = 0$ case is a very good approximation for the small barrier cases. This gives a strong theoretical justification to the usual $m = 0$ assumption in real option theory.

3 Influence of volatility

We now proceed to show that the main result of the theory, that is that uncertainty has a negative effect on investment, remains true with a barrier. As Table 1 illustrates, x^* is an increasing function of volatility σ.

Proposition 3 *The exercise value x^* increases as a function of volatility σ*

I admit the lemma stating that the real option value is convex, that is $f''(x) > 0$ for all x between m and x^*.

Consider a given volatility level σ, and let x^* be the corresponding optimal exercise value and $f(x)$ the real option value function. With volatility $\sigma + h$, let $x^\#$ be the optimal exercise value and $e(x)$ the corresponding real option value function. Assuming that $x^\# < x^*$, the goal is to demonstrate that $h < 0$.

Consider the function $d(x) = f(x) - e(x)$ defined and infinitely smooth on $[m, x^{\#}]$, with $d(m) = 0$.

Consider the set $S = \{x, \forall z \in [x, x^{\#}], d'(z) \leq 0\}$.

Since e satisfies the smooth pasting condition $e'(x^{\#}) = 1$, then $d'(x^{\#}) = f'(x^{\#}) - 1$. Since $f'(x^*) = 1$, the lemma $f'' > 0$ implies that $f'(x^{\#}) < 1$. Therefore $d'(x^{\#}) < 0$. This shows that S is not empty. Let y be its infimum, $y = \inf S$.

This point y is clearly a local maximum, so that $d'(y) = 0$ and $d''(y) \leq 0$. Moreover, $d(y) > 0$ because $d(y) > d(x^{\#}) = f(x^{\#}) - e(x^{\#}) = f(x^{\#}) - (x^{\#} - 1) > 0$.

The function d satisfies the following differential equation since f and g are solution of the Bellman equation 1:

$$2\rho d(x) + 2x(\delta - \rho)d'(x) - x^2\sigma^2 d''(x) + (h + 2\sigma)hx^2 g''(x) = 0$$

Applying the equation at y yields:

$$h(h + 2\sigma)y^2 g''(y) = -2\rho d(y) + x^2\sigma^2 d''(y)$$

Given that $d(y) > 0$ and $d''(y) < 0$, the right hand side is negative. Since (lemma) g'' is positive, we conclude that h cannot be positive.

Q.E.D.

Remark that since the exercise value x^* increases, it has a limit when volatility σ goes to infinity. It can be shown that this limit is actually infinite. The proof only sketched here considers the solution z^* of $z = h_{\beta_1,\beta_2,m}(z)$. It then shows that $z^* < x^*$, and shows that $z^* \to +\infty$ using the explicit formula for z.

Admitting that the optimal level x^* as a function of σ always has a derivative, i.e. the implicit function theorem apply, proposition 3 states that this derivative is positive.

The analysis presented above characterized the variations of the optimal investment threshold $x^* = V^*/I$ to the volatility and barrier parameters. The focus now switches to global expected properties of the decision problem, assuming that investment occurs at the optimal level x^*. Two aspects are considered in this section: the probability that investing will ever occur and the expected decision time.

It is a property of the diffusion process representing x that in the long run, either it has hit x^* and investment has occurred, or it has hit m and the real option has been abandoned.

The probability that a Brownian motion X with drift μ and variance parameter σ starting at level X_0, with $a < X_0 < b$ reaches b before a can be found in classical texts such as [Dixit, 1993, page 64 formula 6.4] or [Karlin and Taylor, 1981, page 205].

The corresponding probability for the geometric Brownian motion x used in the investment model is straightforward to derive: Ito's lemma says that $X = \log(x)$ follows a geometric Brownian motion with drift parameter $\mu = \rho - \delta - \sigma^2/2$ and variance parameter σ, so setting $a = \log(m)$, $b = \log(x^*)$, and $X_0 = \log(x_0)$ gives the result.

This yields the probability of investing someday, that is the probability of hitting x^* before hitting m. When $m > 0$, using $\lambda = 1 + 2(\delta - \rho)/\sigma^2$ as defined equation (2), the probability is:

$$q(x) = \frac{x^\lambda - m^\lambda}{(x^*)^\lambda - m^\lambda} \tag{6}$$

The probability of investing someday when the lower barrier is 0 and the starting point is x can be found by making $m \to 0$. The result depends upon the sign of λ, which can be discussed with μ simply: If $\mu > 0$, then the probability of investing is 1, else it is $(x/x^*)^\lambda$. This points out to a qualitative difference and similarity between the $m = 0$ and $m > 0$ case:

A difference is that with the barrier decision occurs in finite time, whereas with $m = 0$ the process may becomes infinitesimally small forever. Another difference is that when $m > 0$ investment is never certain, even when $\mu > 0$.

But there is a similarity in that when $\sigma^2/2$ is greater than $\rho - \delta$, there is a non-zero probability that investment never occurs even without the barrier assumption. In a high stochasticity regime, the investment probability is less than unity even without a barrier.

We now turn to another proposition that confirms the economic intuition that generally, uncertainty depresses investment.

Proposition 4 *Assuming that the trend $\rho - \delta$ is positive, the probability of investing decreases when volatility σ increases. The limit when $\sigma \to \infty$ is zero.*

The limit of investment probability is zero when $\sigma \to +\infty$ without ambiguity because $x^* \to +\infty$ as we remarked above, and $\lambda \to 1$. The problem is monotonicity. In the sequel, I assume that initially the intrinsic value of the project is just equals to its cost, i.e. $x = 1$.

Since x^* depends upon σ, define η such that $\partial_\sigma x^* = 2x^*\eta$. Then the total derivative $\mathrm{d}q/\mathrm{d}\sigma = \partial_\sigma x^* \, \partial_{x^*} q + \partial_\sigma q$ is:

$$2\frac{m^\lambda((x^*)^\lambda - 1)(\lambda - 1)\log(m) + (m^\lambda - 1)(x^*)^\lambda(\eta\lambda\sigma + (1-\lambda)\log(x^*))}{\sigma(m^\lambda - x^{*\lambda})^2}$$

This expression has the sign of the numerator, which can be rewritten as:

$$(1-\lambda)(m^\lambda - 1)(x^*)^\lambda \log(x^*) + (\lambda - 1)((x^*)^\lambda - 1)m^\lambda \log(m) + (m^\lambda - 1)(x^*)^\lambda \eta\lambda\sigma$$

Assuming that the trend $\rho - \delta$ is positive means that $1 - \lambda > 0$. Dividing by $-(1 - \lambda)(m^\lambda - 1)((x^*)^\lambda - 1)$ does not changes the sign of the expression that becomes:

$$\frac{m^\lambda \log(m)}{m^\lambda - 1} - \frac{(x^*)^\lambda \log(x^*)}{(x^*)^\lambda - 1} - \frac{(x^*)^\lambda \eta\lambda\sigma}{(1-\lambda)((x^*)^\lambda - 1)}$$

Let us study separately the sign of the last term. We know that $\eta > 0$ from proposition 3, that $1 - \lambda > 0$, $\sigma > 0$ and $(x^*)^\lambda > 0$. Remains $\lambda/((x^*)^\lambda - 1)$, which is positive regardless of the sign of λ since $x^* > 1$. Overall, that last term is positive.

Focus now on the difference between the first two terms, written as:

$$D = 1/\lambda\big(\phi(m^\lambda) - \phi((x^*)^\lambda)\big)$$

	σ =0.2	σ =0.4	σ =0.6
m=0	0.4 (0%)	0.28 (0%)	0.19 (0%)
m=0.2	0.39 (-2%)	0.26 (-8%)	0.17 (-10%)
m=0.4	0.37 (-9%)	0.23 (-19%)	0.15 (-20%)
m=0.6	0.32 (-20%)	0.19 (-31%)	0.13 (-32%)
m=0.8	0.25 (-38%)	0.14 (-49%)	0.1 (-49%)

Table 2: Sensitivity of the probability of investing (someday) to the barrier and to volatility

With the auxiliary function $\phi(x) = x \log(x)/(x-1)$. Deriving ϕ show that it's increasing. Since $m < 1 < x^*$, when $\lambda > 0$ we have $m^\lambda < (x^*)^\lambda$ therefore D is negative. When $\lambda < 0$ the inequality is reversed, but because of the $1/\lambda$ factor, D is still negative.

We conclude that $\mathrm{d}q/\mathrm{d}\sigma$ is negative. The probability of investing someday decrease when volatility increases.

Q.E.D.

It seems economically intuitive to ask that the probability of ever investing decreases when the barrier m increases. This question is open. The problem is that when the barrier increases, the ceiling x^* decreases, so the net effect on the probability of hitting the ceiling rather than the barrier is ambiguous. However, numerical simulations presented table 2 suggests that the probability of investing (someday) does decrease as m increases.

4 Variations of the probability of investing

We now turn to the consequences of an important remark: volatility does not only have an effect on the critical level x^*, it also influences the speed at which a decision can be reached. Consequently:

Proposition 5 *The expected life time is not always monotonous with σ.*

The expected exit time of the geometric Brownian motion x can be derived using the formula for $X = log(x)$ found in the same books by Dixit [1993] or Karlin and Taylor [1981]. The expression is complicated and need not to be shown here, since a counter-example is enough to prove the proposition. This is plot figure 2, with $\rho = 0.04$, $\delta = 0.03$ and $m = 0.4$.

The economic sense of this curve is the following. When variance is zero, the expected lifetime is simply given by a "duration = distance / speed" formula. When σ increases a bit, while $\sigma^2/2$ remains small in front of $\rho - \delta$, this puts the exercise price x^* further, and consequently it takes longer to reach it. When σ increases further, the probability of hitting the barrier fast becomes dominant.

This idea that uncertainty affects the speed is important because it question the idea that uncertainty depresses investment, as supported by results up to proposition 4.

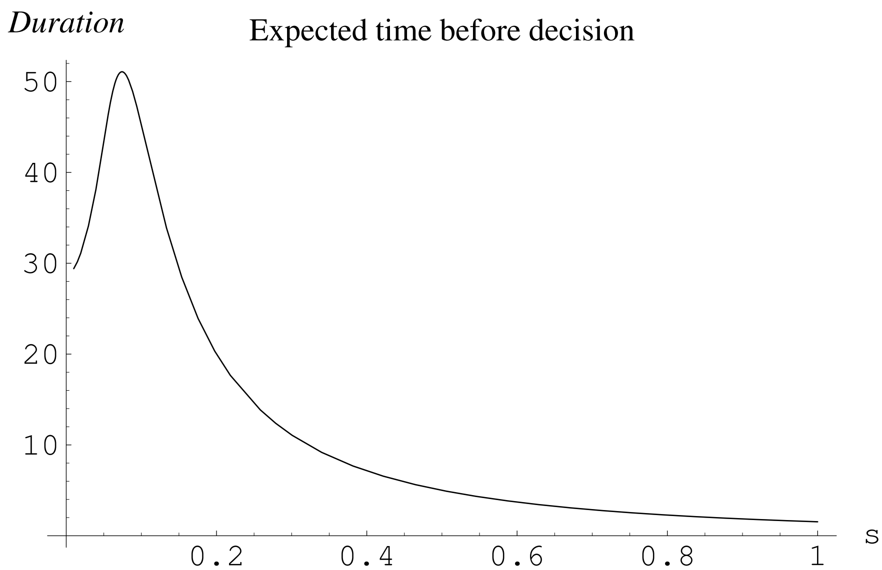

Figure 2: The expected time before decision as a function of volatility σ. ($\rho = 0.04$, $\delta = 0.03$, $m = 0.3$)

Consider for example the no-drift $\rho = \delta$ case, that is an investment with stationary expected value. Assuming that the starting point $V(t = 0)$ is below the cost I, if uncertainty σ is also zero then nothing changes and investment never occurs. Now if uncertainty parameter $\sigma > 0$, there is a non-zero positive probability that investment will occur at some point in time. The investment probability increased from zero when $\sigma = 0$ to some positive number when $\sigma > 0$. At least initially, uncertainty has a positive effect on investment probability. This shows:

Proposition 6 *The probability of having invested at a given date T is not always monotonous with σ, even when $m > 0$.*

This remark that uncertainty does not always play against investment was made by Sarkar [2000]. The original motivation of this paper was that having a barrier V_m would cancel out this effect, because increasing uncertainty would increase the probability of loosing the option faster than the probability of investing. But the above demonstration applies even when there is a non-zero barrier.

It is therefore possible to exhibit an example showing that the probability of having invested at a given date T is not always monotonous with σ even when $m > 0$. The main difficulty is that the probability cannot be expressed in term of usual functions. the interest of the sequel is to give a controlled numerical approximation of the investment probability density at a given date T.

The probability of investing between t and $t+\mathrm{d}t$ is the probability that the diffusion $\log(x)$ exits between t and $t + dt$, conditioned on the exit being in $b = \log(x^*)$ and the diffusion not having touched $a = \log(m)$ before. Since $\log(x)$ is a Brownian motion with drift, according to [Borodin and Salminen, 1996, p. 233], that probability density is:

$$p(t) = \exp\big((b-x)\frac{\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2}t\big)\ ss_{\frac{x-a}{\sigma^2},\frac{b-a}{\sigma^2}}(t) \tag{7}$$

where (op. cit., p. 451):

$$ss_{u,v}(t) = \sum_{k=-\infty}^{k=+\infty} \frac{v-u+2kv}{\sqrt{2\pi}t^{3/2}} \exp\big(-\frac{(v-u+2kv)^2}{2t}\big)$$

Let u_k be the general term of this series. In order to control numerical approximations, introduce $n_k = 2k - u/v$ and examine the ratio of two consecutive terms:

$$r_k = u_k/u_{k-1} = (1 + \frac{1}{n_k - 1})\ \exp\big(-\frac{2v^2 n_k}{t}\big)$$

This ratio is decreasing with k, that is $k > K \Rightarrow r_k < r_K$. Beyond any K chosen large enough so that $n_K > 1$, the series is positive and majorated by a geometric series of reason r_K, therefore $\sum_{k=K+1}^{k=+\infty} u_k < u_K r_K/(1-r_K)$. On the other side, provided that $n_K < -1$, we have $k < -K \Rightarrow r_k > r_{-K}$ therefore $|\sum_{k=-\infty}^{k=-K-1} u_k| < u_{-K}/(r_{-K}-1)$. This insures that when taking $U_K = \sum_{k=-K}^{k=K} u_k$ as an approximation for ss, the maximum absolute error will be $\epsilon(t) = u_K r_K/(1-r_K) - u_{-K}/(r_{-K}-1)$.

Remark that in order to have $r_k < 2\exp(-4A)$, it is enough to choose k such that $n_k > 2$ and $v^2 n_k/t > 2A$. In the geometric Brownian motion of the model, $v = \log(x^*/m)/\sigma^2$, so the later condition writes $n_k > 2At\sigma^4/\log^2(x^*/m)$. Since $u = \log(x_0/m)/\sigma^2$, choosing:

$$K > At\sigma^4/\log^2(x^*/m) + 1/2\log(x_0/m)/\log(x^*/m)$$

is sufficient. It also insures that $r_{-K} > 0.5\exp(4A)$.

That shows that the quality of a finite approximation depends mainly upon m. The closer is m to zero, less terms are needed. It is indeed the case that when $m = 0$, keeping only the central term in the series gives an exact result. To check this, compare the u_0 with [Borodin and Salminen, 1996, p. 223 formula 2.0.2]. When m is close to 1 and the trend is negative, then we know that approximating ss by U will go bad since $\log[x^*/m]$ goes as close to 0 as wanted. When the trend is positive, this does not happens.

Let $P(T)$ be the cumulative probability of exit at the upper boundary before date T, that is the integral of the density $p(t)$ between zero and T. When using U instead of ss, the error on $P(T)$ will be less than:

$$E = \sum_{t=1}^{t=T} \exp\big((b-x)\frac{\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2}t\big)\epsilon(t)dt \tag{8}$$

Figure 3 shows the numerical approximation of the probability of investing, keeping $K = 3$ terms on each side. This is with $T = 5$ years, $\rho = 0.04$, $\delta = 0.03$,$m = 0.3$ and σ between 0.2 and 0.6. With these parameters in equation 8, the algorithmic error E as defined in equation 8 appears to be less than 10^{-30}, that is less than the numerical errors introduced by floating-point computations.

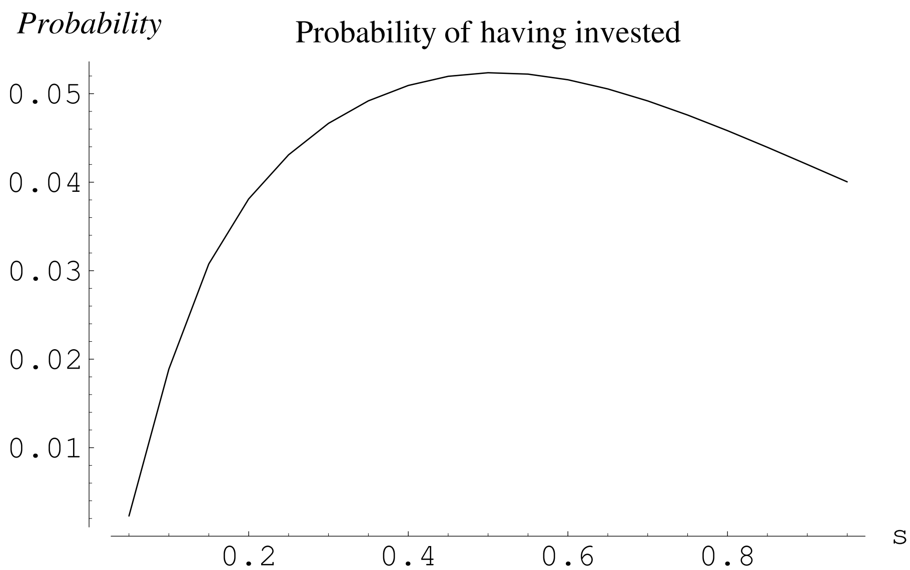

Figure 3: This shows that when σ varies the probability of having invested after $T = 5$ years go through a maximum. It is reached here ($\rho = 0.04$, $\delta = 0.03$, $m = 0.3$) when variance σ is around 0.5.

5 Concluding remarks

To restate the results so far with respect to the barrier and to uncertainty:

The absorbing barrier reduces the level of return V^* required to invest, so that it is optimal to invest sooner when the risk of loosing the opportunity is greater. Table 1 showed that this effect is all the more important that the barrier is close to the investment cost I and that uncertainty is large. On the contrary, when the barrier is close to zero the effect disappears smoothly.

The uncertainty parameter increases the action level V^*, so that it is optimal to invest later when volatility is high. This explains why, under a positive trend assumption, the probability of investing decreases (and the probability of hitting the barrier increases) when volatility increases. If one look only at the outcome expected after a given duration, uncertainty can either increase or decrease the probability of investment.

The model as it is continuously generalizes the basic barrier-free real option model. Several reasons justified the re-visit with the non-zero barrier:

The first motive to explore the $V_m > 0$ assumption is that we do not believe that the value of the investment V can become infinitesimally close to zero. In some real-world cases, the lower absorbing barrier may be crucial to the saliency of the model.

Second, the classical assumption leads sometimes to results that are hard to understand if one is not used to mathematical diffusions theory. An example of this is the infinite expected time to invest, which occurs as soon as variance parameter $\sigma^2/2$

becomes large enough in front of the trend parameter $\rho - \delta$. With a barrier, decision always occur in finite expected time.

Third, the classical assumption $V_m = 0$ is at heart a technical convenience, not an empirical fact. For this reason it is necessary to examine the robustness of results to the alternative assumption.

Another critical aspect of this model is the absorbing nature of the barrier. The model with a reflecting barrier instead would also have some economic interest. It too could continuously generalize the basic model. The reflecting barrier represents the idea of going back to the drawing board, re-shaping from scratch the investment under consideration. In some ways, it is the contrary of irreversibility, and therefore one could expect opposite results in such a model.

However this paper's focused on the decision with the double irreversibility effect, which Ha-Duong [1998] previously discussed in an environmental economics framework. The fact that NASDAQ de-list stocks under one dollar — usually sealing a stock's fate doing so — shows an example of an absorbing barrier relevant for venture capitalists, but the model may also be relevant to other investment issues. When sitting any large business, the need to move-in early to pre-empt the market and lock competitors out should certainly play a role.

References

Borodin and Salminen. *Handbook of Brownian Motion - Facts and Formulae*. Birkhauser, 1996.

Rothschild Brock and Stiglitz. Stochastic capital theory. In G. R. Feiwel, editor, *Joan Robinson and Modern Economic Theory*, chapter 20, pages 591–632. New York University Press, 1989. Revision of NBER tech. paper 23, May 1982.

Avinash Dixit. *The Art of Smooth Pasting*, volume 55 of *Fundamentals of Pure and Applied Economics (ISSN 0191 1708), Stochastic Method in Economic Analysis section*. Harwood Academic Publisher, 1993. ISBN 3-7186-5384-2.

Avinash K. Dixit and Robert S. Pindyck. *Investment under uncertainty*. Princeton University Press, Princeton, New Jersey, 1994.

Bin Gao, Jing zhi Huang, and Marti Subrahmanyam. The valuation of american barrier options using the decomposition technique. *Journal of Economic Dynamics and Control*, 24(11-12):1783–1827, October 2000.

Minh Ha-Duong. Quasi-option value and climate policy choices. *Energy Economics*, 20:599–620, 1998.

Espen Gaarder Haug. Closed form valuation of american barrier options. author is at ehaug@paloma.com, February 10 2000.

Ioannis Karatzas and Hui Wang. A barrier option of american type. ik@math.columbia.edu and wanghui@stat.columbia.edu, April 4 2000.

Karlin and Taylor. *A Second Course in Stochastic Processes*. Academic Press, 1981.

D. R. Rich. *The Mathematical Foundations of Barrier Option Pricing Theory*, volume 7, pages 267–311. 1994.

Sudipto Sarkar. On the investment-uncertainty relationship in a real options model. *Journal of Economic Dynamics and Control*, 24:219–225, 2000.

R. Zvan, K. R. Vetzal, and P. A. Forsyth. PDE methods for pricing barrier options. *Journal of Economic Dynamics and Control*, 24:1563–1590, 2000.